Algebraic Curves homework 1, due 9/28 in class

1. Prove that when $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$ -linearly independent, then

1. $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is discrete.
2. $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is Hausdorff.
3. $\mathbb{C} \to \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a covering map.

2. Let V be a complex vector space of dimension n, with $\mathbb{C}$-basis $e_1, \dots, e_n$, and $T\colon V \to V$ is a $\mathbb{C}$-linear transformation. Suppose T has matrix representation $X = A + Bi$ where $A, B \in M_n(\mathbb{R})$ under (complex) basis $e_1, \dots, e_n$. Prove

1. $e_1, \dots, e_n, ie_1, \dots, ie_n$ is an $\mathbb{R}$-basis of V.
2. T has matrix
$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$
under the $\mathbb{R}$-basis above when T is viewed as an $\mathbb{R}$-linear transformation.
3. $$\det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = |\det X|^2.$$

3. Complete the proof of implicit function theorem: Assume $f(z, w)$ is holomorphic w.r.t $(z, w) \in U \subset \mathbb{C}^2$, and $\frac{\partial f}{\partial z} \neq 0$ for all $(z, w) \in U$, and
$$\{f = 0\} \cap U = \{(g(w), w) \mid w \in D\}$$
for some open subset $D \subset \mathbb{C}$.

Prove g is holomorphic w.r.t w by showing $\frac{\partial g}{\partial \bar{w}} = 0$.

4. Let $x_1, \dots, x_n$ be distinct points on $\mathbb{C}$ and
$$f(x, y) = y^d - (x - x_1) \cdots (x - x_n).$$
Prove that $C = \{f(x, y) = 0\}$ defines a Riemann surface in $\mathbb{C}^2$. (Question to think about: what is the topological shape of C?)

5.(Extra excersice. You get extra credit for this exercise) Prove the implicit function theorem for several variables.

Let $f_1(z_1, \dots z_n), \cdots, f_m(z_1, \dots, z_n)$ be holomorphic functions w.r.t $(z_1, \dots, z_n) \in U \subset \mathbb{C}^n$, where $m \leq n$. Assume $(a_1, \dots, a_n) \in U$ satisfies $f(a_1, \dots, a_n) = 0$ and
$$\left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$$

is nondegenerate at $(a_1, \ldots, a_n)$.

Then there exists a neighborhood V of $(a_1, \ldots, a_n)$ and holomorphic functions

$$g_1(z_{m+1}, \ldots, z_n), \ldots, g_m(z_{m+1}, \ldots, z_n)$$

defined on a neighborhood D of $(a_{m+1}, \ldots, a_n) \in \mathbb{C}^{n-m}$ such that

$$\{f = 0\} \cap V = \{(g_1(z_{m+1}, \ldots, z_n), \ldots, g_m(z_{m+1}, \ldots, z_n), z_{m+1}, \cdots, z_n) \mid (z_{m+1}, \ldots, z_n) \in D\}.$$